Universal behavior in the parametric evolution of chaotic saddles

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Chaotic saddles are nonattracting dynamical invariant sets that physically lead to transient chaos. As a system parameter changes, chaotic saddles can evolve via an infinite number of homoclinic or heteroclinic tangencies of their stable and unstable manifolds. Based on previous numerical evidence and a rigorous analysis of a class of representative models, we show that dynamical invariants such as the topological entropy and the fractal dimension of chaotic saddles obey a universal behavior: they exhibit a devil-staircase characteristic as a function of the system parameter. [S1063-651X(99)01605-0]

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Chaotic saddles are nonattracting dynamical invariant sets in the phase space of nonlinear systems [1–3]. A trajectory starting from a random initial condition in a phase-space region containing a chaotic saddle typically stays near the saddle exhibiting a chaoticlike dynamics for a finite amount of time before exiting the region eventually and asymptoting to a final state (usually not chaotic). Chaos in this case is only transient. Physically, chaotic saddles lead to observable phenomena such as chaotic scattering [4], fractal basin boundaries [5], and passive particle advection in open hydrodynamical flows [6]. Mathematically, chaotic saddles are closed, bounded, and invariant sets having a dense orbit. They are the soul of chaotic dynamics [7].

A central problem in chaotic dynamics concerns how a dynamical invariant set evolves or bifurcates as a system parameter changes. If the invariant set is attracting, i.e., a stable periodic orbit or a chaotic attractor, qualitative changes of the set can be conveniently studied by the bifurcation diagram [8]. It is known that the bifurcation of an attracting set can exhibit universal behaviors, such as those observed in successive period-doubling bifurcations of stable periodic orbits [9]. However, a bifurcation diagram, which deals with attractors, cannot reveal qualitative changes of chaotic saddles because they are nonattracting. Hence, one has to rely on more quantitative measures such as the behavior of various dynamical invariants of chaotic saddles to characterize their parametric evolution. A question is then, what is the universal behavior characterizing the invariants associated with the evolution of chaotic saddles?

The aim of this paper is to present a universal behavior, a behavior that is highly nontrivial, governing the evolution of chaotic saddles that arise in dissipative chaotic systems or in open Hamiltonian systems. Specifically, let p be a bifurcation parameter and let $[p_1,p_2]$, where $p_2>p_1$, be a parameter interval in which there is a chaotic saddle. Assume that the chaotic saddle evolves as p is increased from p_1 to p_2. Dynamically, the universal feature of such an evolution can be characterized by an infinite number of homoclinic or heteroclinic tangencies and subsequent crossings of the stable and unstable manifolds associated with the saddle. Then, if one measures dynamical invariants such as the topological entropy or the fractal dimension, one finds that the change of these invariants as a function of the system parameter p typically exhibits a *devil-staircase* type of behavior. That is, in the parameter interval $[p_1,p_2]$, the dynamical invariants remain constant or change smoothly in almost all the intervals, but, at a set of an infinite number of parameter values of Lebesgue measure zero, they change *abruptly*. To establish universality of the devil-staircase behavior, we study an analyzable model that captures the essential dynamical features involved in the evolution of chaotic saddles: the sequence of tangencies between the stable and unstable manifolds.

Before we proceed with our analysis, we wish to point out the following three physical contexts in which the evolution of chaotic saddle and the devil-staircase behavior are relevant [10].

(i) Crisis in chaotic scattering. Chaotic scattering is a manifestation of chaotic saddles in open Hamiltonian systems [4]. A crisis in chaotic scattering [11] is characterized by the collision and interaction of two previously isolated chaotic saddles via a complicated sequence of intersections of their stable and unstable manifolds. A physical consequence is that chaotic scattering is characteristically enhanced throughout the crisis in the sense that an infinite number of new possibilities for scattering trajectories is being created due to the tangencies of stable and unstable manifolds. Dynamically, the fractal dimension of the chaotic saddle increases through the crisis. It was conjectured, but never shown in Refs. [11], that the fractal-dimension function of the chaotic saddle versus a system parameter throughout the crisis exhibits a devil-staircase behavior.

(ii) Channel capacity in communicating with chaos. It was demonstrated recently that a chaotic system can be manipulated, via arbitrarily small time-dependent perturbations, to generate controlled chaotic orbits whose symbolic representation corresponds to the digital representation of a desirable message [12]. A central issue in any digital communication strategy is to select a proper coding scheme by which arbitrary messages can be encoded into the transmitting signal. It was argued that in general, a coding scheme generates chaotic trajectories that live on one of the infinite number of nonattracting chaotic saddles embedded in the chaotic attrac-

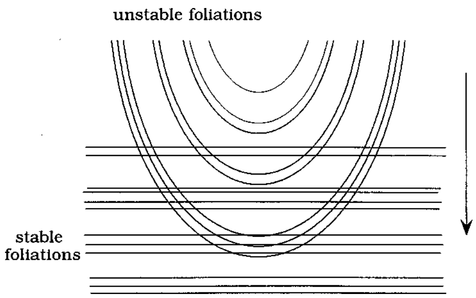

FIG. 1. Schematic illustration of interaction between stable and unstable foliations of a chaotic saddle. Parametric evolution of the system corresponds to unstable foliations' moving downwards.

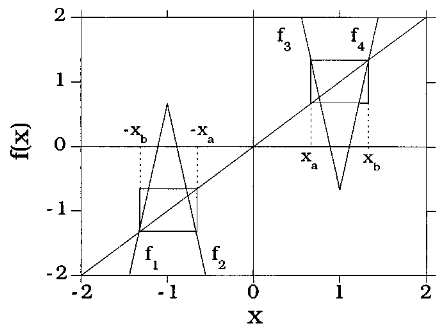

FIG. 2. Illustration of the model Eq. (1).

tor [13]. A relevant question is how much information the system can encode and transmit. A quantitative measure of the amount of information is the *channel capacity* [14], which is equivalent to the topological entropy of the chaotic set that is utilized for encoding messages. Since a coding scheme makes use of only an invariant subset embedded in the attractor, and since the topological entropy of the subset cannot be greater than that of the attractor, the channel capacity in any practical communication scheme employing a code must be less than or equal to that which would be produced in the ideal situation where the full attractor is utilized for encoding messages. In Refs. [13], it was demonstrated that the channel-capacity function versus a parameter characterizing the chaotic saddle typically exhibits a devil-staircase behavior as a result of the coding.

(iii) Evolution of chaotic saddles after crisis. Crisis in dissipative dynamical systems is an event that converts a chaotic attractor into a chaotic saddle [1] as a system parameter changes through a critical value. In a typical nonlinear system, a crisis is induced by the collision of a chaotic attractor with the boundary of its own basin. Since the attractor lies in the closure of its unstable manifold, and since the basin boundary is the stable manifold of a saddle periodic orbit on the boundary, the collision can be characterized as a homoclinic or heteroclinic tangency. As the parameter increases further through the crisis, an infinite number of tangencies occurs because both the stable and unstable manifolds of the invariant chaotic saddle after the crisis possess a fractal structure. Consequently, the chaotic saddle keeps evolving after the crisis. It was found numerically that the topological entropy of the chaotic saddle after crisis typically exhibits a nondecreasing devil-staircase type of behavior [15].

We now give a qualitative argument for the devil-staircase behavior. Consider an invariant chaotic saddle in the two-dimensional plane. Since the saddle has a horseshoe-like structure, both the stable and unstable foliations are fractals, as shown schematically in Fig. 1, where the horizontal lines denote segments of the stable manifold and the curved ones are those of the unstable manifold. To characterize evolution of the saddle, assume that, as a system parameter changes, the unstable foliations move downwards across the stable foliations. At a generic parameter value, some unstable manifold becomes tangent to the stable one—a homoclinic or a heteroclinic tangency. Dynamically, an infinite number of unstable periodic orbits is created about such a tangency [16]. Thus, we expect the dynamical invariants, such as the topological entropy that measures the abundance of unstable periodic orbits, to increase abruptly at the tangencies. Due to the fractal structure of the stable and unstable foliations, such tangencies occur at a dense set of parameter values of Lebesgue measure zero. In any parameter subintervals where there is no tangency, the dynamical invariants remain constant or change smoothly, since the topology of the stable and unstable manifolds remains unchanged. Overall, in a parameter inverval containing both the first and the last tangencies, we expect to see the values of dynamical invariants increase abruptly at each tangency value, while they remain constant in any subintervals in between the tangency parameter values.

The essential ingredient leading to a devil-staircase behavior in the dynamical invariants of a chaotic saddle, which is independent of the physics of any specific model system, is the tangencies between the stable and unstable foliations of the saddle. We thus seek to construct a model that captures this essential feature, yet the model should be simple enough so that a rigorous understanding can be obtained. To the extent that the model is free of any feature specific to physical systems exhibiting a devil-staircase behavior, we can regard predictions of the model as *universal*. We consider the following one-dimensional model:

$$f_b(x)=\begin{cases} -a|x+1|+b & \text{for } x\leqslant 0,\\ a|x-1|-b & \text{for } x>0, \end{cases} \tag{1}$$

where $x\in\mathbf{R}$, and a and b are parameters $(b>-1)$. The map is schematically shown in Fig. 2, where the four branches of the map are labeled by $f_1(x)$ and $f_4(x)$ (positive slopes) and $f_2(x)$ and $f_3(x)$ (negative slopes). The map is invariant under the following symmetrical operations: $x\rightarrow -x$ and $f_b(x)\rightarrow -f_b(x)$. For small values of a, the map exhibits bounded attractors, while for large a values, almost all initial conditions, except a set of Lebesgue measure zero, asymptote to either ∞ or $-\infty$. Since we are interested in modeling transient chaos, we fix a at a reasonably large value and investigate the dynamical behavior of the map as b is increased from zero. There are two intervals in which the chaotic saddles live: $A_+\equiv[-x_b,-x_a]$ and $A_-\equiv[x_a,x_b]$, where $x_a=(a-b-2)/(a-1)$ and $x_b=(a+b)/(a-1)$, which are determined by $a(x_b-1)-b=x_b$ and $a(1-x_a)-b=x_b$. To assure that almost all initial conditions asymptote to $\pm\infty$, we require that $x_a>0$, or equivalently, $a\geqslant(b+2)$. Tangencies occur when the critical point and its high-order iterates of, say, the left-hand side branch, touch the edges of the box defined by x_a and x_b of the right-hand side branch, and vice versa.

Note that Eq. (1) is piecewise linear and the absolute value of the derivative is constant: $|f_b'(x)|=a$ for any x. Hence, the natural invariant measure μ_* [17] covers uniformly the invariant set S. Moreover, it coincides with the Parry measure (maximal entropy measure) and the Sinai-Ruelle-Bowen (SRB) measure [18,19]. Thus the topological entropy h_T, the Kolmogorov-Sinai (KS) metric entropy $h_{\rm KS}$, and the generalized Renyi entropies h_q of the system are identical: $h_T=h_{\rm KS}=h_q$. In an analogous way, the constant slope of the map stipulates that the generalized dimensions (including the capacity D_0, the information dimension D_1, and the correlation dimension D_2 [20]) are all equal. Our system is similar to the system analyzed by Bohr and Rand [18], in which a relation between the information dimension and the KS entropy is given by $D_1=h_{\rm KS}/\ln a$. This relationship, corresponding to the Kaplan-Yorke conjecture [21], is also valid for our model. Thus, if the fractal dimension exhibits a devil-staircase behavior, so does the topological entropy, and vice versa. For concreteness, in the analyses that follow, we set $a=6$. For this value of a, all tangencies occur for $b\in[b_1,b_2]=[2/3,3/2]$. Our aim is to provide a rigorous calculation for the behavior of the topological entropy and the fractal dimension for b in this range.

The topological entropy of a chaotic system is the asymptotic rate of growth of the number of periodic orbits with respect to the length of the period [19]. Recently it was proposed that for one-dimensional maps, the topological entropy could be computed by averaging the number of preimages with respect to the maximal entropy measure μ_* [22]. Consider a one-dimensional mixing system f: $X\to X$, where the function f is piecewise monotone and continuous on N branches. Its topological entropy is then given by

$$h_T=\ln\int_X P(x)d\mu_*(x), \tag{2}$$

where $P(x)$: $X\to\{0,1,2,...,N\}$ represents the number of preimages of f at the point x, restricted to the support of μ_*. For Eq. (1), there are only two preimages for $b<b_1$, which gives $h_T=\ln 2$. For $b>b_2$, there are four preimages so that the topological entropy is $h_T=2\ln 2$. Equation (2) is also applicable to cases where $b\in(b_1,b_2)$, since the measure of maximal entropy μ_* is uniformly distributed over the invariant set and may be approximated by an iteration procedure. Figure 3(a) shows the branches of that map in $x>0$. To take into account the coupling with the left-hand side branch of the system, we use two auxiliary functions $f_5(x)=|f_3(x)|$ and $f_6(x)=|f_4(x)|$. Point b in Fig. 3(a) splits the invariant set S^+ into two parts: P_4 and P_2. For $x<b$ belonging to S^+ there exist four preimages, while for $x\in P_2$ there are only two. Making use of Eq. (2), we obtain

$$h_T=\ln\left(4\int_{x_a}^{b}d\mu_*(x)+2\int_{b}^{x_b}d\mu_*(x)\right)=\ln 2+\ln(1+M), \tag{3}$$

where the relative weight M of the subset P_4 depends on b and is given by

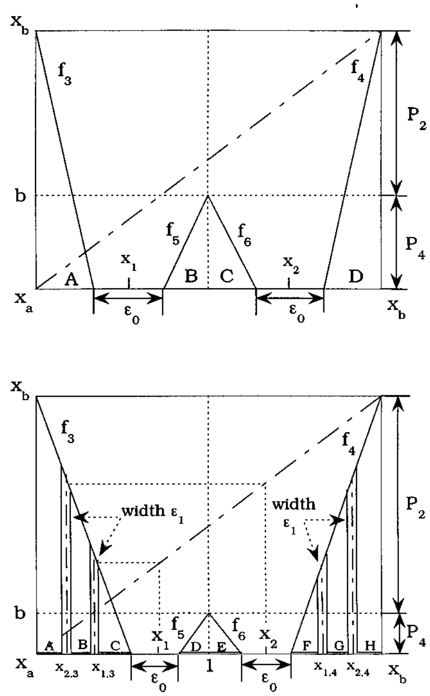

FIG. 3. (a) A detailed view of the right branch of Eq. (1); (b) the four preimages of the two gaps of width ϵ_0 in (a).

$$M=\int_{x_a}^{b}d\mu_*(x). \tag{4}$$

From the relation $D_1=h_{\rm KS}/\ln a$, we obtain an *exact* result for the fractal dimension D of the chaotic saddle S for Eq. (1) at $a=6$:

$$D=\frac{\ln 2+\ln(1+M)}{\ln 6}. \tag{5}$$

To compute the fractal dimension D, it is thus necessary to compute M, which can be obtained by successive approximations to the fractal measure $d\mu_*$. To obtain the crudest (zeroth-order) approximation $D^{(0)}$, we use the interval S_0 instead of the fractal set S^+. This is equivalent to using in Eq. (4) the Lebesgue measure dx in place of the fractal measure $d\mu_*$. The relative weight of P_4 is then approximated by the ratio $M^0(b)=(b-x_a)/(x_b-x_a)$, which when substituted into Eq. (5) yields

$$D^{(0)}(b)=\begin{cases}\dfrac{\ln 2}{\ln 6} & \text{for } b<2/3,\\[2ex] \dfrac{\ln 2+\ln[(4b-1)/(b+1)]}{\ln 6} & \text{for } b\in[2/3,3/2]\\[2ex] \dfrac{2\ln 2}{\ln 6} & \text{for } b>3/2.\end{cases} \tag{6}$$

This function is shown by the dashed curve in Fig. 4. A better approximation, $D^{(1)}$, is obtained by taking into account the two main gaps in the set S^+, as shown in Fig. 4 by the thin solid curve. Discontinuity in $D^{(1)}$ occurs if the bifurcation parameter b coincides with the edges of the gaps g_1

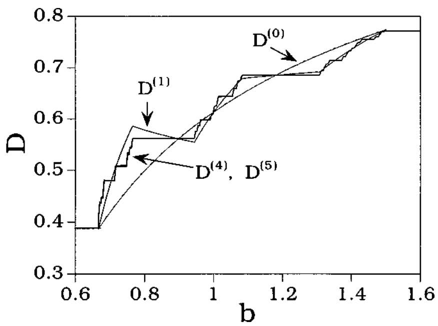

FIG. 4. Devil-staircase of the fractal dimension D as a function of b. Narrow dashed and solid lines represent continuous approximations of $D^{(0)}(b)$ and $D^{(1)}(b)$, respectively, while the thick line represents $D^{(5)}(b)$.

and g_2. Every gap in the invariant fractal set S corresponds to a specific plateau in the function $D(b)$, which is a devil staircase. To visualize this structure in a more transparent way, we show in Fig. 3(b) a more precise sketch of the invariant set S^+, in which the secondary gaps are indicated. These secondary gaps are the preimages of the primary gaps: $f_b^{-1}(g_1)$ and $f_b^{-1}(g_2)$. There are four secondary gaps for the value of b used in this graph: for larger values of b two or four new gaps appear in the central part D and E. These secondary gaps have the width $\varepsilon_1 = \varepsilon_0/a$ and are centered at the preimages of the points x_1 and x_2. In Fig. 3(b), the symbol $x_{2,3}$ denotes the point $f_3^{-1}(x_2)$, etc. The set S_{i+1} can then be constructed by removing from S_i the preimages of all its gaps. Defining μ_i to be a uniform measure on the set S_i, we can compute the number $M^{(i)} = \int_{x_a}^{b} d\mu_i / \int_{x_a}^{x_b} d\mu_i$. The sequence of the measures μ_i converges in a weak sense to μ_*, so the sequence of integrals $M^{(i)}$ converges to M, defined as an integral over a fractal measure.

Due to the large contraction factor $(a=6)$, the convergence of $M^{(i)}$ is fast: the numerically obtained fifth-order approximation of the fractal dimension $D^{(5)}(b)$ (represented in Fig. 4 by a thick line) is already hardly distinguishable from the fourth-order one $D^{(4)}(b)$. Vertical lines indicate positions of the primary plateaus. Observe that the dependence $D(b)$ between them, say for $b \in (17/13, 3/2)$, is similar to the dependence in the entire interval (b_1, b_2). Equation (6) thus allows for a simple interpretation of the existence of the dimension plateaus and hence the devil-staircase behavior: if the parameter b sweeps the gaps of the fractal set S, integral (4) remains constant, and so is the dimension D.

In summary, we make use of the concept of integration over a fractal measure to obtain analytically, to arbitrarily high order approximations, the fractal dimension and the topological entropy of chaotic saddles evolving under parameter change. Our paradigm captures the essential features of evolution of chaotic saddles: an infinite number of tangencies between the stable and unstable foliations. As such, the devil-staircase behavior of dynamical invariants is a characteristic feature in physical phenomena involving the evolution of nonattracting chaotic saddles.

We stress that although the model utilized in our analysis is a one-dimensional map with symmetry, it captures the essential dynamical feature of the parametric evolution of chaotic saddles in higher-dimensional systems: the infinite number of tangencies between the stable and the unstable manifolds of the saddle. This feature is responsible for the devil-staircase behavior in the dynamical invariants of the chaotic saddle. Symmetry in our one-dimensional model is not an essential ingredient for the devil-staircase behavior: we use it only to facilitate the *analytic derivation*. In fact, there is no apparent symmetry in higher-dimensional maps such as the Hénon map, yet *numerical evidence* for the devil-staircase behavior is clearly present in the dynamical invariants of the chaotic saddles [10,13,15]. In so far as the stable and the unstable foliations of a chaotic set are fractals, their interaction during the parametric evolution of the set will result in the devil-staircase behavior. This is independent of any specific features, such as symmetry, of the system.

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